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STATISTICAL ANALYSIS OF ASTRONOMICAL UNIT ESTIMATES

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PREFACE

The discrepancy between existing estimates of the astronomical unit far exceeds the previously accepted probable errors. This RAND Memorandum discusses three aspects of statistical theory, each of which may be relevant to resolving the discrepancy. It should be of interest to anyone (astronomer or statistician) who estimates solar system constants. It was supported by the National Aeronautics and Space Administration under Contract No. NASr-21 (04). One of the authors, Allan Marcus, was a consultant to The RAND Corporation.

SUMMARY

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In determining the astronomical unit, (1) any neglect of the possible correlation of errors may result in an underestimate of the standard deviation of the a.u. estimate; (2) an estimate of the parameters of primary interest with small variance may be based on a least-squares fit using reasonable assumed values of the additional parameters as constants, rather than including them in the fit; and (3) the variance of an a.u. estimate may be estimated validly from a single observation of a related phenomenon (e.g., distance from Earth to Venus), provided that the variance of that particular observation is known.

Author

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CONTENTS

STATISTICAL ANALYSIS OF ASTRONOMICAL UNIT ESTIMATES

1. INTRODUCTION

This Memorandum discusses three separate aspects of statistical theory, each relevant to the problem of how data on astronomical unit (a.u.) determinations may be used with statistical efficiency.

The discrepancy between Rabe's estimate [3] of the a.u. (1.495256×10^8 km) and the JPL estimate [4] of the a.u. (1.495988×10^8 km) is over 10 times the cited probable error of Rabe's estimate and about 300 times the cited probable error of the JPL estimate. Assuming no systematic errors in either estimate, the only logical reconciliation of the two estimates would be, by careful examination, to discover that the estimated probable errors are indeed too small. The first section of this Memorandum indicates a possible source of this underestimate--the effect of correlation upon the variance of a sample average. A common example of this is when separate a.u. estimates are made and then averaged from the data of the same experiment.

The second section explores the question of whether or not additional parameters, of which we have good prior estimates, should be treated as unknowns and, along with the parameters of direct interest, be estimated via least squares. Such treatment will unquestionably lead to a smaller residual sum of squares, but the resulting estimates of the parameters of primary interest may vary more widely than would estimates of these parameters that are based on least squares but use our prior estimates for the values of the additional parameters.

Note, however, that a systematic error or bias is introduced when an inexact prior estimate of a parameter of incidental interest is used in a least-squares analysis. Thus there may be a trade-off between magnitude of systematic error and magnitude of probable error that should determine which mode of analysis is appropriate in a given situation. The results of Eckstein [1] may be viewed in this light. He shows that 25.6% of the discrepancy between Rabe's and JPL's estimates of the solar parallax (and hence 26.5% of the discrepancy between their a.u. estimates) are due to their using different values of some incidental solar system parameters. In addition, the probable error of the estimate of the solar parallax decreased from Rabe's $0\overset{''}{.}00039$ to $0\overset{''}{.}00008$. This, then, is an example of a situation where the introduction of a better value for an incidental parameter decreases the magnitudes of both the systematic and probable errors.

The third section shows how to obtain the variance of an estimate of a parameter based on a single observation. This technique is then applied to an idealized version of the JPL experiment. Because the JPL estimate of the a.u. is an average of several estimates, each of our calculated variances will differ from the variance of the JPL estimate. However, the variances tabulated here are combined in the manner explained in Section 2 (assuming no correlation between successive estimates) to obtain an order-of-magnitude indication of the variance of the JPL a.u. estimate which is consistent with their cited value of the probable error.

2. EFFECT OF CORRELATION ON VARIANCE OF A SAMPLE MEAN

For brevity of nomenclature, we shall extend the use of the word experiment to include any exercise in which observations are made on a physical phenomenon. Using this same word, therefore, we can refer both to Rabe's observations on the orbit of Eros, a natural phenomenon, and to JPL's observations on a radar probe of Venus, a man-made phenomenon.

Measurements from such an "experiment" are usually imprecise, in that the observed value is composed of the true value, which one hoped to observe, plus errors. These errors are usually classified as being either systematic errors, which are due to a definite cause and act according to a definite law, or random errors, which show no regularities and are indistinguishable from random numbers drawn from some probability distribution whose mean is zero. We shall assume that the only errors present in our observations are random errors. The probability distribution of these errors is usually the normal distribution with parameter σ, the standard deviation.

Usually, many observations are made of the same phenomenon. If we let ξ be the true value, we can express the observations as $X_1, \ldots, X_n$, where $X_i = \xi + U_i$ and U_i is a random error. Since $E[U_i]$, the expected value of U_i, is zero for all i, we see that an unbiased estimate of ξ is

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

The standard assumption made about the U_i's is that they are independent and identically distributed. If so, the variance of U_i, $V[U_i]$, equals σ^2 for all i. Then

$$V[\overline{X}] = \frac{\sigma^2}{n} .$$

If the U_i's are independent but not identically distributed, so that $V[U_i] = \sigma_i^2$, then

$$V[\overline{X}] = \frac{\sum_{i=1}^{n} \sigma_i^2}{n^2} .$$

Finally, if the U_i's are also correlated, with ρ_{ij} the correlation between U_i and U_j, then

$$V[\overline{X}] = \frac{\sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho_{ij}\, \sigma_i\, \sigma_j}{n^2} .$$

We are being so explicit because if $\overline{X}$ is used as an estimate of ξ, the standard deviation of the estimate is $\sqrt{V[\overline{X}]}$. When the estimate is normally distributed (as is $\overline{X}$ in the above cases), the "probable error" π is defined as

$$\Pr\{\overline{X} - \pi < \xi < \overline{X} + \pi\} = \frac{1}{2} .$$

Thus $\pi = .674\sqrt{V[\overline{X}]}$. In this paper, we shall be interested only in $V[\overline{X}]$ and its square root, and not in π. We shall therefore avoid the mistake of too many experimenters who loosely use the term "probable error" to mean $\sqrt{V[\overline{X}]}$, rather than π.

Another confusion occurs when some phenomenon is said to be "known to within δ." This may state the limits of the experiment's observational precision, but not the probable error or the standard deviation of U. The last two measure the reproducibility of the observed value if the experiment is repeated, a measurement that of course depends on, but is not synonymous with, the limits of observational precision.

The confusion among these three different measures of error may cause some of the discrepancy in the estimates of error for the a.u. estimates.

One usually uses the quantity

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n(n-1)}$$

as an estimate of $V[\overline{X}]$. The expected value of S^2 is

$$E[S^2] = \frac{\sum_{i=1}^{n} E[X_i^2] - n(E[\overline{X}]^2)}{n(n-1)}$$

$$= \frac{\sum_{i=1}^{n} \sigma_i^2 + n\xi^2 - nV[\overline{X}] - n\xi^2}{n(n-1)}$$

$$= \frac{\sum_{i=1}^{n} \sigma_i^2 (1 - \frac{n}{n^2})}{n(n-1)} - \frac{2n \sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho_{ij}\, \sigma_i\, \sigma_j}{n(n-1)n^2}$$

$$= \frac{\sum_{i=1}^{n} \sigma_i^2}{n^2} - \frac{2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho_{ij}\, \sigma_i\, \sigma_j}{n^2(n-1)}$$

$$\neq V[\overline{X}] ,$$

except when $\rho_{ij} = 0$ for all i, j. Thus, only if the observations are uncorrelated, is the "mean-square-deviation divided by n" an unbiased estimate of the variance of $\overline{X}$.

In the simple case when $\sigma_i^2 = \sigma^2$ for all i and $\rho_{ij} = \rho$ for all i, j, then

$$V[\overline{X}] = \frac{n\sigma^2 + n(n-1)\rho\sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n} \{1 + (n-1)\rho\} ,$$

where ρ must exceed $-1/(n-1)$. In any event, if $\rho > 0$, which is true for most physical situations, then $V[\overline{X}] > \sigma^2/n$. Thus a simple division of σ^2 (or an estimate of σ^2) by n will yield an underestimate of the variance of $\overline{X}$.

In a more complicated, but more realistic case, where the X_i's are a time sequence of observations, one might postulate that the correlation between X_i and X_j falls off as the time between making the observations X_i and X_j increases. One such correlation model is given by $\rho_{ij} = \rho^{|i-j|}$. When $\sigma_i^2 = \sigma^2$ for all i, then in this model

$$V[\overline{X}] = \frac{\sigma^2}{n}\frac{(1+\rho)}{(1-\rho)} - \frac{2\sigma^2}{n^2}\frac{\rho(1-\rho^n)}{(1-\rho)^2} .$$

For large n, the second term is negligible. One thus sees that here, too, when $\rho > 0$ a simple division of σ^2 (or an estimate of σ^2) by n will yield an underestimate of the variance of $\overline{X}$, by a factor of $(1+\rho)/(1-\rho)$.

From this, we see that by neglecting the possible correlation of errors we may underestimate the error in the true value's estimate.

3. EFFECT OF FITTING ADDITIONAL PARAMETERS IN LEAST-SQUARES ANALYSIS

In the previous section, we discussed experiments in which only one parameter is to be estimated. In actuality, though, the measurable phenomenon ξ is a function of many parameters and the experimenter usually attempts to use his observations on ξ to estimate all these parameters. In doing this, he typically employs some variant of least-squares analysis, and finds the set of parameters that minimizes some (perhaps weighted) residual sum of squares. Certainly this is the best set of estimates of the parameters that characterize the ξ's. It does not follow, however, that any particular estimate in this set is the best estimate (the one with minimum variance) of its corresponding parameter.

It may be better, by using these independent estimates of some of the set's parameters, to find least-squares estimates of the remaining parameters of interest. This should give a larger residual sum of squares but a smaller variance of a given parameter's estimate.

To illustrate our contention, consider the following simple example. Let

$$\xi = \beta\eta_1 + \gamma\eta_2 ,$$

where ξ, η_1, and η_2 are measurable phenomena and β

and γ are unknown parameters. Suppose we observe n independent triples $(X_1, \eta_{11}, \eta_{21}), \ldots, (X_n, \eta_{1n}, \eta_{2n})$, where $X_i = \xi_i + U_i$ and the η's are observed without error. We assume that the U_i's are independent and identically distributed with mean zero and variance σ^2. Both β and γ are parameters characterizing ξ. But supposing the parameter of particular interest to be β, we shall contrast two estimation procedures--one which assumes a value of γ and estimates only β, and another which estimates both β and γ.

Define

$$S_X = \sum_{i=1}^{n} X_i^2 ,$$

$$S_{Xj} = \sum_{i=1}^{n} X_i \eta_{ji} , \quad \text{and}$$

$$S_{jk} = \sum_{i=1}^{n} \eta_{ji} \eta_{ki}$$

for $j, k = 1, 2$. Now suppose we assume that the value of γ is some constant γ_o, a universally accepted value, say. Then the least-squares estimator of β is

$$\hat{\beta}(\gamma_o) = \frac{S_{X1} - \gamma_o S_{12}}{S_{11}} .$$

This estimator has expected value

$$E\hat{\beta}(\gamma_o) = \beta + (\gamma - \gamma_o) \frac{S_{12}}{S_{11}} ,$$

so that its bias is $(\gamma - \gamma_o)\ S_{12}/S_{11}$.

An unbiased estimate of σ^2 is

$$\hat{\sigma}^2(\gamma_o) = \frac{(S_X S_{11} - S_{X1}^2) - 2\gamma_o(S_{11}S_{X2} - S_{12}S_{X1}) + \gamma_o^2(S_{22}S_{11} - S_{12}^2)}{(n-1)S_{11}},$$

and the variance of $\hat{\beta}(\gamma_o)$ is estimated by

$$\hat{V}[\hat{\beta}(\gamma_o)] = \frac{\hat{\sigma}^2(\gamma_o)}{S_{11}} .$$

Let us contrast this with the case in which both β and γ are estimated from the same experiment via least squares. In this case the least-squares estimators of β and γ are

$$\hat{\beta} = \frac{S_{X1}\ S_{22} - S_{X2}\ S_{12}}{S_{11}\ S_{22} - S_{12}^2} ,$$

$$\hat{\gamma} = \frac{S_{X2}\ S_{11} - S_{X1}\ S_{12}}{S_{11}\ S_{22} - S_{12}^2} ,$$

which are both unbiased. An unbiased estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{S_{XX} + \hat{\beta}^2 S_{11} + \hat{\gamma}^2 S_{22} + 2\hat{\beta}\hat{\gamma} S_{12} - 2\hat{\beta} S_{X1} - 2\hat{\gamma} S_{X2}}{(n-2)} ,$$

and the variance of $\hat{\beta}$ is estimated by

$$\hat{V}[\hat{\beta}] = \frac{\hat{\sigma}^2\ S_{22}}{S_{11}S_{22} - S_{12}^2}$$

We illustrate the difference between $\hat{V}[\hat{\beta}(\gamma_o)]$ and $\hat{V}[\hat{\beta}]$ by the following table, based on data taken from Example 20.1 of Reference [3]. In this example, $\hat{\gamma} = .662249$. Since both $\hat{V}[\hat{\beta}(\gamma_o)]$ and the absolute value of the bias of $\hat{\beta}(\gamma_o)$ are symmetric functions of ϵ, we tabulate them only for $\epsilon > 0$, and, instead of studying the effect of varying γ_o on $\hat{V}[\hat{\beta}(\gamma_o)]$, we shall for convenience express γ_o as $\gamma_o = .662249 + \epsilon$ and tabulate $\hat{V}[\hat{\beta}(\gamma_o)]$ as a function of ϵ. We also tabulate the bias of $\hat{\beta}(\gamma_o)$ on the assumption that $\gamma = .662249$.

ϵ	Bias of $\hat{\beta}(\gamma_o)$	$\hat{V}[\hat{\beta}(\gamma_o)]$
.008	.0048	.01166
.018	.0109	.01180
.028	.0169	.01205
.038	.0230	.01241
.048	.0290	.01288
.058	.0351	.01347

Since

$$\min_{\gamma} \min_{\beta} \sum_{i=1}^{n} (X_i - \beta\eta_{1i} - \gamma\eta_{2i})^2$$

$$= \sum_{i=1}^{n} (X_i - \hat{\beta}\eta_{1i} - \hat{\gamma}\eta_{2i})^2 \quad ,$$

and since

$$\min_{\beta} \sum_{i=1}^{n} (X_i - \beta\eta_{1i} - \gamma_o\eta_{2i})^2$$

$$= \sum_{i=1}^{n} (X_i - \hat{\beta}(\gamma_o)\eta_{1i} - \gamma_o\eta_{2i})^2 ,$$

we see that $\hat{\beta}(\hat{\gamma}) = \hat{\beta}$. We find for this example that $\hat{V}[\hat{\beta}] = \hat{V}[\hat{\beta}(\hat{\gamma})] = .01338$. Comparing this with the tabulated values, we see that if we do not estimate γ from the data but in fact use an accepted value γ_o of γ within $\pm$ .055 of $\hat{\gamma}$, then our estimate $\hat{\beta}(\gamma_o)$ has a smaller variance than that based on multiple regression. Of course, part of this is a reflection of the fact that in computing $V[\hat{\beta}]$ we treat our concomitant estimate of γ, $\hat{\gamma}$, as a random variable, whereas in computing $V[\hat{\beta}(\gamma_o)]$ we treat our estimate of γ, γ_o, as a constant.

The independent estimate γ_o of γ may be a random variable, an estimate of γ based on an independent random sample, with mean γ and variance τ^2. In that case,

$$V[\hat{\beta}(\gamma_o)] = \frac{\sigma^2(\gamma_o)}{S_{11}} + \tau^2(\frac{S_{12}}{S_{11}})^2$$

In our example, $(S_{12}/S_{11})^2 = 18.67$; so for

$$\tau^2 \leq \frac{.01338 - \frac{\sigma^2(\gamma_o)}{S_{11}}}{18.67} = \tau_U^2 ,$$

$\hat{\beta}(\gamma_o)$ is a better estimate of β than is $\hat{\beta}$. For the above values of ϵ, the following are upper bounds on τ, the standard deviation of γ_o.

ϵ	τ_U
.008	.0096
.018	.0092
.028	.0084
.038	.0072
.048	.0052

4. ESTIMATES BASED ON INVERSES OF PHYSICAL RELATIONSHIPS

All unreplicated experiments for determining only the a.u., $a_\oplus$, can be described in the following way. The true value, ξ, is related to $a_\oplus$ by the known physical relation $\xi = f(a_\oplus)$; the observed value X is the sum of ξ and U, where U is a random observational error uncorrelated with ξ whose expected value is zero. One then estimates $a_\oplus$ by $\hat{a}_\oplus = f^{-1}(X)$. Since X is a random variable whose variance equals σ^2, the variance of U, the estimate $\hat{a}_\oplus$ is also a random variable, whose variance we shall now approximate.

If we expand $f^{-1}(X)$ in Taylor series around $\xi = f(a_\oplus)$ and retain first-order terms, we obtain

$$\hat{a}_\oplus \simeq f^{-1}(\xi) + \left.\frac{df^{-1}(X)}{dX}\right|_{X = \xi} (X - \xi)$$

$$= a_\oplus + \frac{U}{f'(a_\oplus)} ,$$

where $f'(a_\oplus)$ is the derivative of f evaluated at $a_\oplus$, the true a.u. Then

$$V[\hat{a}_\oplus] \simeq \frac{\sigma^2}{[f'(a_\oplus)]^2}$$

Notice that, based on this expansion, we have $E[\hat{a}_\oplus] \simeq a_\oplus$. If we look closely at the inversion, though, by examining the second-order terms, we find that

$$\hat{a}_{\oplus} \simeq a_{\oplus} + \frac{U}{f'(a_{\oplus})} - \frac{f''(a_{\oplus})}{2\xi^2 f'(a_{\oplus})} U^2 ,$$

so that

$$E[\hat{a}_{\oplus}] \simeq a_{\oplus} - \frac{f''(a_{\oplus})}{2[f'(a_{\oplus})]^3} \sigma^2$$

$$= a_{\oplus} + b(a_{\oplus}) ,$$

where $f''(a_{\oplus})$ is the second derivative of f evaluated at $a_{\oplus}$. Thus using $f^{-1}(X)$ as $\hat{a}_{\oplus}$ yields a biased estimate of $a_{\oplus}$, and this bias, $b(a_{\oplus})$, can be evaluated.

Keeping this in mind, let us look at the JPL estimates of $a_{\oplus}$ based on data derived from radar signals reflected from Venus. Possible measurable phenomena, at any point in time, are range and radial velocity.

The physical relation, f, between ξ and the a.u., $a_{\oplus}$, is simplest to express if we assume that the orbits of Earth and Venus are concentric, circular, and coplanar. Since these orbits have low eccentricities and inclinations, the approximation is reasonable over the relatively brief period of observation of the JPL experiment.

Let us measure time from the Earth-Venus conjunction (so that $t = 0$ at conjunction). We define $a_{♀}$ as the semi-major axis of the orbit of Venus, and $T_{\oplus}$ and $T_{♀}$

as the orbital periods of Earth and of Venus. We assume that the quantity $\delta = T_\oplus/T_♀$ is known precisely, as is $k = 2\pi\sqrt{GM_\odot}$ (or equivalently, Gauss's constant). Then, since $a_♀ = \delta^{-2/3}\, a_\oplus$, the aforementioned measurable phenomena are functions only of $a_\oplus$. The three we shall be concerned with are:

(1) range: $$a_\oplus\{[\cos(kt/a_\oplus^{3/2}) - \delta^{-2/3}\cos(\delta kt/a_\oplus^{3/2})]^2 + [\sin(kt/a_\oplus^{3/2}) - \delta^{-2/3}\sin(\delta kt/a_\oplus^{3/2})]^2\}^{1/2} = f_1(a_\oplus);$$

(2) radial velocity: $$\{k\delta^{-2/3}(\delta-1)a_\oplus^{1/2}\sin[(\delta-1)kt/a_\oplus^{3/2}]\}/f_1(a_\oplus) = f_2(a_\oplus);$$

(3) radial acceleration: $$-f_2^2(a_\oplus)/f_1(a_\oplus) + \{k^2\delta^{2/3}(\delta-1)^2\cos[(\delta-1)kt\, a_\oplus^{-3/2}]\}/a_\oplus\, f_1(a_\oplus) = f_3(a_\oplus).$$

We suppress for the moment the dependence of f_1 and f_2 on t, because we are only concerned here with the variance of a single determination of $\hat{a}_\oplus$.

We see that

$$f_1'(a_\oplus) = [f_1(a_\oplus) - \tfrac{3}{2}t\; f_2(a_\oplus)]/a_\oplus$$

$$f_2'(a_\oplus) = [f_2(a_\oplus) - 3t\; f_3(a_\oplus) - 3t\; f_2^2(a_\oplus)/f_1(a_\oplus)]/2a_\oplus\ .$$

Let X_1 be the observed range of Venus at time t, with U_1 its associated random error. Let X_2 be the observed radial velocity of Venus at time t, with U_2 its associated random error. Let $\hat{a}^1_\oplus = f_1^{-1}(X_1)$, $\hat{a}^2_\oplus = f_2^{-1}(X_2)$, $\sigma_1^2 = V[U_1]$, and $\sigma_2^2 = V[U_2]$. Then

$$V[\hat{a}^1_\oplus] \simeq \sigma_1^2/[f_1'(a_\oplus)]^2$$

$$V[\hat{a}^2_\oplus] \simeq \sigma_2^2/[f_2'(a_\oplus)]^2 .$$

Reference [6] gives scant indication as to the value of σ_1 and σ_2. Its only statement relating to σ_1 is the following: "The accuracy of the time-of-flight measurement is about 1 millisec in a typical round trip flight time of 3 x 10^5 millisec." Let us interpret this as saying that $\sigma_1 = (1/3)10^{-5}\xi_1$, where ξ_1 is the true range at a particular time.

In the same reference, the only statement relating to σ_2 is that the "accuracy of the velocity measurement is about one part in 10^5." We interpret this as saying that $\sigma_2 = 10^{-5}\ \xi_2$, where ξ_2 is the maximum true doppler shift of the experiment. So interpreted, this statement is misleading, as it suggest that errors in velocity measurement are small, when, in fact, the absolute magnitude of σ_2 is quite great for large doppler shifts.

Reference [4] states that the doppler-frequency counters had a resolution of 1 cycle per second. We have also learned [2] that the estimated standard deviation of the $\dot{R}(t)$'s is 0.14 m/sec. Since

$$\frac{\dot{R}(t)}{2c} = \frac{\Delta f(t)}{f_o} ,$$

where $\Delta f(t)$ is the doppler frequency shift at time t and f_o = 2388 megacycles, we see that

$$\begin{aligned} \sqrt{V[\Delta f(t)]} &= \frac{f_o}{2c} \sqrt{V[\dot{R}(t)]} \\ &= \frac{2.4 \times 10^9 \times .14}{2 \times 3 \times 10^8} \\ &= .56 \text{ cycles,} \end{aligned}$$

which is consistent with the resolution information. We therefore use 1.4×10^{-4} km/sec as σ_2.

Table 1 gives values of $\sqrt{V[\hat{a}^1_\oplus]}$ and of $\sqrt{V[\hat{a}^2_\oplus]}$, as calculated from the above approximations, for various values of t ($t = 0$ being the time of conjunction, roughly April 10.4, 1961).

Table 1

t (days)	Date	$\sqrt{V[\hat{a}^1_\oplus]}$ km	$\sqrt{V[\hat{a}^2_\oplus]}$ km
-11.4	March 30	600	6300
-5.4	April 5	525	12500
-1.4	April 9	500	48000
1.6	April 12	500	40000
5.6	April 16	525	12500
11.6	April 22	600	6300
21.6	May 2	1425	4000

The bias of $\hat{a}^1_\oplus$ is on the order of 10^{-6} km, rendering negligible systematic errors in $\hat{a}^1_\oplus$. Without determining it, we suspect the same for the bias of $\hat{a}^2_\oplus$.

The order of magnitude of $\sqrt{V[\hat{a}^1_\oplus]}$ is consistent with the ± 100 figure given in Reference [4], for if we disregard the May 2 observation as being spuriously high due to the inappropriateness of our assumptions, the standard deviation of the average of the six a.u. determinations (assuming no correlation) is 125.

On the other hand, the standard deviations of the doppler-frequency estimates as given in Table 1 are much larger than those consistent with a 500-km standard deviation of an average. However, since the JPL estimates and standard deviations are based on the eastern and western elongation of Venus, rather than on conjunction data, the cited 500-km (and possibly even the 200-km) standard deviation can be the result of basing the a.u. estimate on enough uncorrelated a.u. determinations with standard deviations of 6000 km or less.

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